Compressing exterior Helmholtz problems: 1D indefinite Laplacian

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Contents

1 Introduction

This script reproduces Example 6.1 in [2]. It computes RKFIT approximants for $f_h(A)\boldsymbol{b}$, where A is a symmetric indefinite matrix corresponding to the discretization of a 1D Laplacian and $f_h(\lambda) = \sqrt{\lambda + (h\lambda/2)^2}$. The RKFIT approximants are compared to the uniform two-interval Zolotarev approximants in [1].

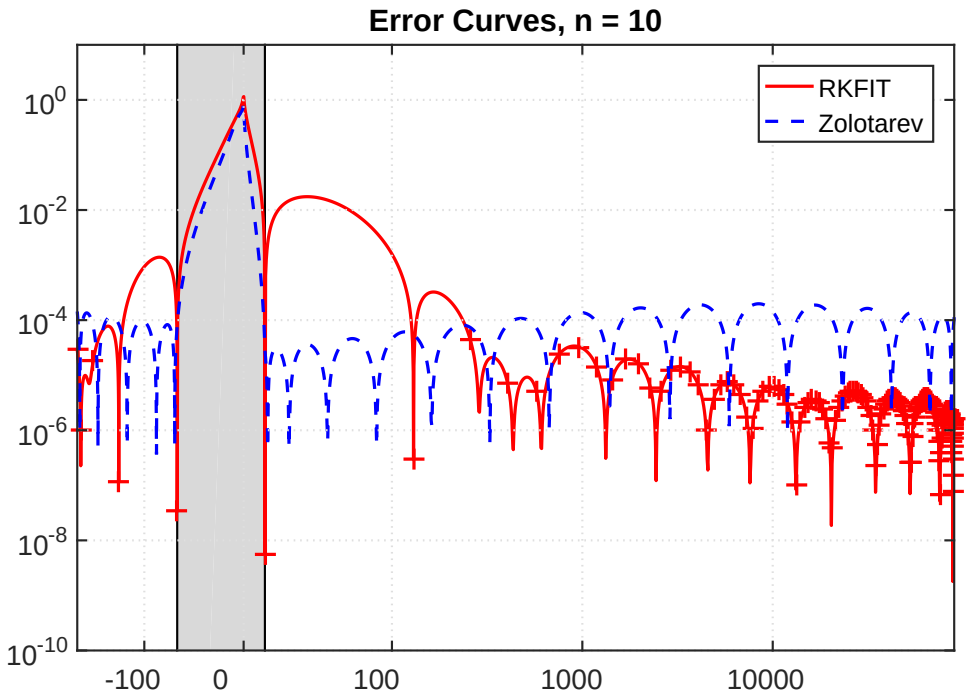
2 The code

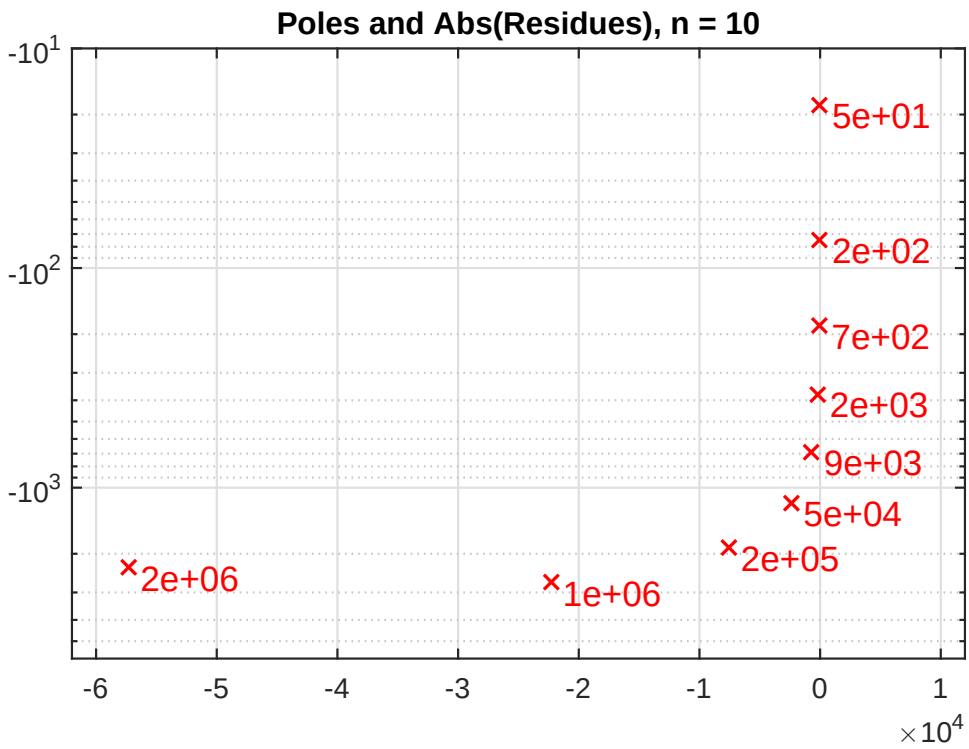
```
N = 150;
h = 1/N; % grid step
k = 15;
L = gallery('tridiag',N); L(1,1) = 1; L(N,N) = 1;
A = 1/h^2*L - k^2*speye(N);
F = sqrtm(full(A) + (h*full(A)/2)^2);

bt = randn(N,1); bt = bt/norm(bt); % training vector
b = randn(N,1); b = b/norm(b);

ee = sort(eig(full(A))).';
a1 = min(ee);
b1 = max(ee(ee<0));
a2 = min(ee(ee>0));
b2 = max(ee);
```

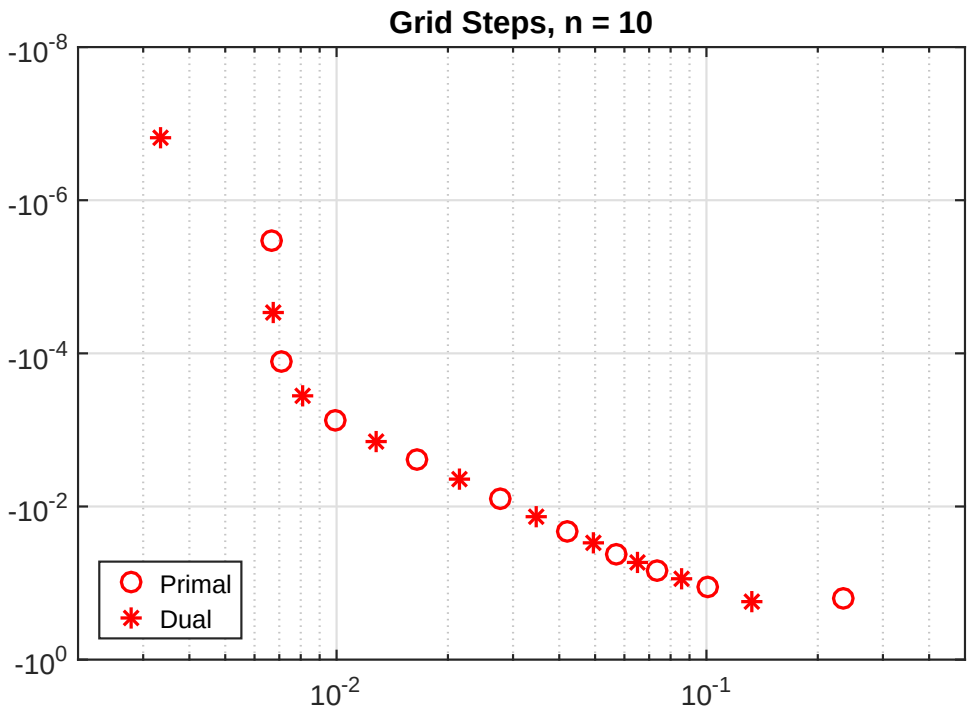
Initialize RKFIT parameters.

```
param = struct;
param.reduction = 0;
param.k = 1; % superdiagonal
param.tol = 0;
param.real = 0;

for m = 1:25,

    % run rkfit with training vector
    param.maxit = 5;
    xi = inf*ones(1,m-1); % take m-1 initial poles
    [xi,ratfun,misfit,out] = rkfit(F,A,bt,xi,param);
    if m==1, err_rkfitt = out.misfit_initial; iter_rkfit = 0;
    else
        [err_rkfitt(m),iter_rkfit(m)] = min(misfit);
        iter_rkfit(m) = find(misfit <= 1.01*min(misfit),1);
    end

    % recompute best ratfun and compute error for vector b
    param.maxit = iter_rkfit(m);
    xi = inf*ones(1,m-1); % take m-1 initial poles
    [xi,ratfun,misfit,out] = rkfit(F,A,bt,xi,param);
    err_rkfit(m) = norm(F*b - ratfun(A,b))/norm(F*b);

    ex = @(x) sqrt(x + (h*x/2).^2);
    zolo = rkfun('sqrt2h',a1,b1,a2,b2,m,h);
    % check matrix-vector error
    err_zolo(m) = norm(F*b - zolo(A,b))/norm(F*b);

    if m == 10, % some plots for m = 10

        figure
        semilogy(NaN); hold on
        lint = util_log2lin([b1,a2],[a1,b1,a2,b2],.1);
        fill([lint(1:2),lint([2,1])], [1e-25,1e-25,1e15,1e15],
           ...
            .85*[1,1,1], 'LineStyle', '-')
        ylim([1e-10,10])
        ax = [ -10.^(5:-1:2) , 0 , 10.^(2:5) ];
        linax = util_log2lin(ax,[a1,b1,a2,b2],.1);
        %labels = num2str(ax(:),'%1.0G');
        set(gca,'XTick',linax,'XTickLabel',ax)
        xx = [ -logspace(log10(-a1),log10(-b1),1000) , linspace(
           b1,a2,200) , ...
            logspace(log10(a2),log10(b2),1000) ];
        xx = union(xx,ee);
        xxt = util_log2lin(xx,[a1,b1,a2,b2],.1);
        eet = util_log2lin(ee,[a1,b1,a2,b2],.1);
        hdl1 = semilogy(xxt,abs(ratfun(xx) - ex(xx)),'r-');
        semilogy(eet,abs(ratfun(ee) - ex(ee)),'r+')
```

```
        hdl2 = semilogy(xxt,abs(zolo(xx) - ex(xx)),'b--');
        legend([hdl1,hdl2],'RKFIT','Zolotarev ')
        xlim([0,1])
        title(['Error Curves, n = ' num2str(m) ])
        grid on
        set(gca,'layer','top')

        % plot residues
        figure
        [resid,xi] = residue(mp(ratfun),2);
        resid = double(resid); xi = double(xi);
        semilogy(xi,'rx')
        axis([-6.2e4,1.2e4,-6e3,-1e1]), hold on
        labels = num2str(abs(resid),'%0.1g');
        hdl = text(real(xi)+1e3, imag(xi)*1.1, labels);
        set(hdl,'Color','r','FontSize',16,'Rotation',0)
        set(hdl(end),'Rotation',0)
        title(['Poles and Abs(Residues), n = ' num2str(m)])
        grid on

        % plot grid steps
        figure
        [grid1,grid2,absterm,cnd,pf] = contfrac(mp(ratfun));
        grid1 = double(grid1); grid2 = double(grid2);
        loglog(grid1,'ro'), hold on
        loglog(grid2,'r*')
        legend('Primal','Dual','Location','SouthWest')
        title(['Grid Steps, n = ' num2str(m)]), grid on
        axis([2e-3,5e-1,-1,-1e-8])
    end
end % for m
```

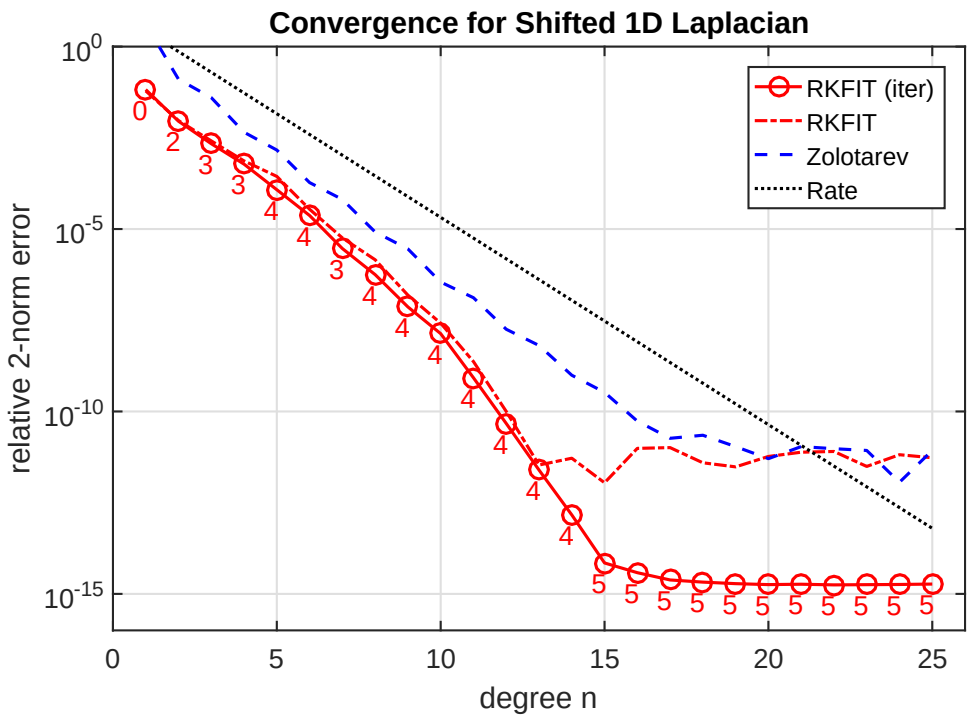







```
figure
semilogy(err_rkfitt,'r-o')
hold on
semilogy(err_rkfit,'r-.')
semilogy(err_zolo,'b--')
rate = exp(-2*pi^2/log((256*a1*b2/a2/b1)));
semilogy(10*rate.^(1:m),'k:')
ylim([1e-16,1])
xlim([0,m+1])
```

```
xlabel('degree n')
ylabel('relative 2-norm error')
legend('RKFIT (iter)','RKFIT','Zolotarev','Rate')
labels = num2str(iter_rkfit(:),'%d');
hdl = text((1:m)-.4,err_rkfitt/10,labels,'horizontal','left','
    vertical','bottom');
set(hdl,'FontSize',13,'Color','r'), grid on
title('Convergence for Shifted 1D Laplacian')
```



3 Conclusions

The main observation is that, due to spectral adaptation, the RKFIT approximants significantly outperform the uniform Zolotarev approximants and yield superlinear convergence in accuracy. Only a small number of RKFIT iterations is required to compute these approximants.

4 Other examples

The other examples in [2] can be reproduced with the following scripts:

Figure 1.2 - an infinite waveguide with two layers

Example 6.2 - constant coefficient and 2D indefinite Laplacian

Example 6.3 - uniform approximation on indefinite interval

Example 7.1 - truly variable-coefficient case with 2D indefinite Laplacian

5 References

[1] V. Druskin, S. Güttel, and L. Knizhnerman. *Near-optimal perfectly matched layers for indefinite Helmholtz problems,* SIAM Rev., 58(1):90–116, 2016.

[2] V. Druskin, S. Güttel, and L. Knizhnerman. *Compressing variable-coefficient exterior Helmholtz problems via RKFIT,* MIMS EPrint 2016.53 (http://eprints.ma.man.ac.uk/2511/), Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2016.