

# A Rational Krylov Toolbox for MATLAB

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## 1 Overview

Thank you for your interest in the Rational Krylov Toolbox (RKToolbox). The RKToolbox is a collection of scientific computing tools based on rational Krylov techniques. The development started in 2013 and the current version 2.3 provides

- an implementation of Ruhe’s rational Krylov sequence method [5, 6], allowing to control various options, including user-defined inner products, exploitation of complex-conjugate shifts, orthogonalization, and rerunning [4],
- algorithms for the implicit and explicit relocation of the poles of a rational Krylov space [3],
- a collection of utility functions, e.g., for solving nonlinear eigenvalue problems,
- an implementation of RKFIT [3, 4], a robust algorithm for rational L2 approximation, including automated degree reduction, and
- the RKFUN class [4] allowing for numerical computations with rational functions, including support for MATLAB Variable Precision Arithmetic and the Advanpix Multiple Precision toolbox [1].

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This guide explains the main functionalities of the toolbox. To run the embedded MATLAB codes the RKToolbox needs to be in MATLAB's search path. For details about the installation we refer to the **Download** section on <http://rktoolbox.org/>.

## 2 Rational Krylov spaces

A rational Krylov space is a linear vector space of rational functions in a matrix times a vector. Let  $A$  be a square matrix of size  $N \times N$ ,  $\mathbf{b}$  an  $N \times 1$  starting vector, and let  $\xi_1, \xi_2, \dots, \xi_m$  be a sequence of complex or infinite *poles* all distinct from the eigenvalues of  $A$ . Then the rational Krylov space of order  $m + 1$  associated with  $A, \mathbf{b}, \xi_j$  is defined as

$$\mathcal{Q}_{m+1}(A, \mathbf{b}, q_m) = q_m(A)^{-1} \text{span}\{\mathbf{b}, A\mathbf{b}, \dots, A^m \mathbf{b}\},$$

where  $q_m(z) = \prod_{j=1, \xi_j \neq \infty}^m (z - \xi_j)$  is the common denominator of the rational functions associated with the rational Krylov space. The rational Krylov method by Ruhe [5, 6] computes an orthonormal basis  $V_{m+1}$  of  $\mathcal{Q}_{m+1}(A, \mathbf{b}, q_m)$ . The basis matrix  $V_{m+1}$  satisfies a rational Arnoldi decomposition of the form

$$AV_{m+1}\underline{K}_m = V_{m+1}\underline{H}_m,$$

where  $(\underline{H}_m, \underline{K}_m)$  is an (unreduced) upper Hessenberg pencil of size  $(m + 1) \times m$ .

Rational Arnoldi decompositions are useful for several purposes. For example, the eigenvalues of the upper  $m \times m$  part of the pencil  $(\underline{H}_m, \underline{K}_m)$  can be excellent approximations to some of  $A$ 's eigenvalues [5, 6]. Other applications include matrix function approximation and rational quadrature, model order reduction, matrix equations, nonlinear eigenproblems, and rational least squares fitting (see below).

## 3 Computing rational Krylov bases

**Relevant functions:** `rat_krylov`, `util_cplxpair`

Let us compute  $V_{m+1}$ ,  $\underline{K}_m$ , and  $\underline{H}_m$  using the `rat_krylov` function, and verify that the outputs satisfy the rational Arnoldi decomposition by computing the relative residual norm  $\|AV_{m+1}\underline{K}_m - V_{m+1}\underline{H}_m\|_2 / \|\underline{H}_m\|_2$ . For  $A$  we take the `tridiag` matrix of size 100 from MATLAB's `gallery`, and  $\mathbf{b} = [1, 0, \dots, 0]^T$ . The  $m = 5$  poles  $\xi_j$  are, in order,  $-1, \infty, -i, 0, i$ .

```
N = 100; % matrix size
A = gallery('tridiag', N);
b = eye(N, 1); % starting vector
xi = [-1, inf, -1i, 0, 1i]; % m = 5 poles
[V, K, H] = rat_krylov(A, b, xi);
resnorm = norm(A*V*K - V*H)/norm(H) % residual check
```

```
resnorm =
    4.1662e-15
```

As some of the poles  $\xi_j$  in this example are complex, the matrices  $V_{m+1}$ ,  $\underline{K}_m$ , and  $\underline{H}_m$  are complex, too:

```
disp([isreal(V), isreal(K), isreal(H)])
```

```
0      0      0
```

However, the poles  $\xi_j$  can be reordered so that complex-conjugate pairs appear next to each other using the function `util_cplxpair`. After reordering the poles, we can call the function `rat_krylov` with the 'real' option, thereby computing a real-valued rational Arnoldi decomposition [5].

```
% Group together poles appearing in complex-conjugate pairs.
xi = util_cplxpair(xi);
[V, K, H] = rat_krylov(A, b, xi, 'real');
resnorm = norm(A*V*K - V*H)/norm(H)
disp([isreal(V), isreal(K), isreal(H)])
```

```
resnorm =
    6.4056e-15
    1      1      1
```

Our implementation `rat_krylov` supports many features not shown in the basic description above.

- It is possible to use matrix pencils  $(A, B)$  instead of a single matrix  $A$ . This leads to decompositions of the form  $AV_{m+1}K_m = BV_{m+1}H_m$ .
- Both the matrix  $A$  and the pencil  $(A, B)$  can be passed either explicitly, or implicitly by providing function handles to perform matrix-vector products and to solve shifted linear systems.
- Non-standard inner products for constructing the orthonormal bases are supported.
- One can choose between CGS and MGS with or without reorthogonalization.
- Iterative refinement for the linear system solves is supported.

For more details type `help rat_krylov`.

## 4 Moving poles of a rational Krylov space

**Relevant functions:** `move_poles_expl`, `move_poles_impl`

There is a direct link between the starting vector  $\mathbf{b}$  and the poles  $\xi_j$  of a rational Krylov space  $\mathcal{Q}_{m+1}$ . A change of the poles  $\xi_j$  to  $\check{\xi}_j$  can be interpreted as a change of the starting vector from  $\mathbf{b}$  to  $\check{\mathbf{b}}$ , and vice versa. Algorithms for moving the poles of a rational Krylov space are described in [3] and implemented in the functions `move_poles_expl` and `move_poles_impl`.

**Example:** Let us move the  $m = 5$  poles  $-1, \infty, -i, 0$ , and  $i$  into  $\check{\xi}_j = -j$ ,  $j = 1, 2, \dots, 5$ .

```
N = 100;
A = gallery('tridiag', N);
b = eye(N, 1);
xi = [-1, inf, -1i, 0, 1i];
[V, K, H] = rat_krylov(A, b, xi);
xi_new = -1:-1:-5;
[KT, HT, QT, ZT] = move_poles_expl(K, H, xi_new);
```

The poles of a rational Krylov space are the eigenvalues of the lower  $m \times m$  part of the pencil  $(\check{H}_m, \check{K}_m)$  in a rational Arnoldi decomposition  $A\check{V}_{m+1}\check{K}_m = \check{V}_{m+1}\check{H}_m$  associated with that space [3]. By transforming a rational Arnoldi decomposition we are therefore effectively moving the poles:

```
VT = V*QT';
resnorm = norm(A*VT*KT - VT*HT)/norm(HT)
moved_poles = util_pencil_poles(HT, KT).'
```

```
resnorm =
    6.8668e-15
moved_poles =
    -1.0000e+00 + 1.0476e-16i
    -5.0000e-01 - 2.1919e-16i
    -3.3333e-01 - 1.7310e-16i
    -2.5000e-01 - 2.7182e-16i
    -2.0000e-01 + 3.3675e-16i
```

## 5 Rational Krylov fitting (RKFIT)

**Relevant function:** rkfit

RKFIT [3, 4] is an iterative Krylov-based algorithm for nonlinear rational approximation. Given two families of  $N \times N$  matrices  $\{F^{[j]}\}_{j=1}^{\ell}$  and  $\{D^{[j]}\}_{j=1}^{\ell}$ , an  $N \times n$  block of vectors  $B$ , and an  $N \times N$  matrix  $A$ , the algorithm seeks a family of rational functions  $\{r^{[j]}\}_{j=1}^{\ell}$  of type  $(m+k, m)$ , all sharing a common denominator  $q_m$ , such that the *relative misfit*

$$\text{misfit} = \sqrt{\frac{\sum_{j=1}^{\ell} \|D^{[j]}[F^{[j]}B - r^{[j]}(A)B]\|_F^2}{\sum_{j=1}^{\ell} \|D^{[j]}F^{[j]}B\|_F^2}} \rightarrow \min$$

is minimal. The matrices  $\{D^{[j]}\}_{j=1}^{\ell}$  are optional, and if not provided  $D^{[j]} = I_N$  is assumed. The algorithm takes an initial guess for  $q_m$  and iteratively tries to improve it by relocating the poles of a rational Krylov space.

We now show on a simple example how to use the `rkfit` function. Consider again the tridiagonal matrix  $A$  and the vector  $\mathbf{b}$  from above and let  $F = A^{1/2}$ .

```
N = 100;
A = gallery('tridiag', N);
b = eye(N, 1);
F = sqrtm(full(A));
exact = F*b;
```

Now let us find a rational function  $r_m(z)$  of type  $(m, m)$  with  $m = 10$  such that  $\|F\mathbf{b} - r_m(A)\mathbf{b}\|_2 / \|F\mathbf{b}\|_2$  is small. The function `rkfit` requires an input vector of  $m$  initial poles and then tries to return an improved set of poles. If we had no clue about where to place the initial poles we can easily set them all to infinity. In the following we run RKFIT for at most 15 iterations and aim at relative misfit  $\|F\mathbf{b} - r_m(A)\mathbf{b}\|_2 / \|F\mathbf{b}\|_2$  below  $10^{-10}$ . We display the error after each iteration.

```
[xi, ratfun, misfit] = rkfit(F, A, b, ...
                           repmat(1, 1, 10), ...
                           15, 1e-10, 'real');

disp(misfit)
```

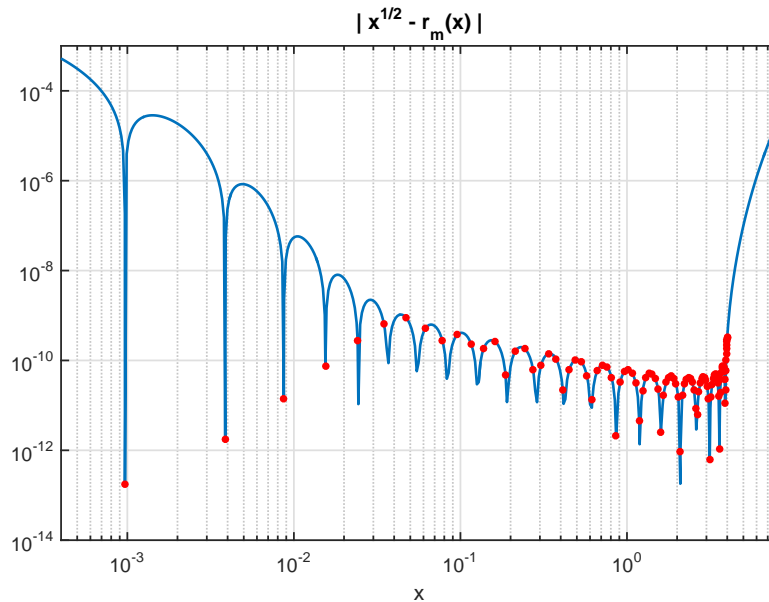
```
7.8082e-07    1.4769e-10    4.6371e-11
```

The rational function  $r_m(A)\mathbf{b}$  of type (10,10) approximates  $A^{1/2}\mathbf{b}$  to about 10 decimal places. A useful output of `rkfit` is the RKFUN object `ratfun` representing the rational function  $r_m$ . It can be used, for example, to evaluate  $r_m(z)$ :

- `ratfun(A,v)` evaluates  $r_m(A)\mathbf{v}$  as a matrix function times a vector,
- `ratfun(A,V)` evaluates  $r_m(A)V$  as a matrix function times a matrix, e.g., setting  $V = I$  as the identity matrix will return the full matrix function  $r_m(A)$ , or
- `ratfun(z)` evaluates  $r_m(z)$  as a scalar function in the complex plane.

Here is a plot of the error  $|x^{1/2} - r_m(x)|$  over the spectral interval of  $A$  (approximately  $[0, 4]$ ), together with the values at the eigenvalues of  $A$ :

```
figure
ee = eig(full(A)).';
xx = sort([logspace(-4.3, 1, 500) , ee]);
loglog(xx,abs(sqrt(xx) - ratfun(xx))); hold on
loglog(ee,abs(sqrt(ee) - ratfun(ee)), 'r.', 'markers', 15)
axis([4e-4, 8, 1e-14, 1e-3]); xlabel('x'); grid on
title('| x^{1/2} - r_m(x) |', 'interpreter','tex')
```



As expected the rational function  $r_m(z)$  is a good approximation of the square root over  $[0, 4]$ . It is, however, not a uniform approximation because we are approximately minimizing the 2-norm error on the eigenvalues of  $A$ , and moreover we are implicitly using a weight function given by the components of  $\mathbf{b}$  in  $A$ 's eigenvector basis.

Additional features of RKFIT are listed below.

- An automated degree reduction procedure [4, Section 4] is implemented; it takes place if a relative misfit below tolerance is achieved, unless deactivated.
- Nondiagonal rational approximants are supported; can be specified via an additional `param` structure.
- Utility functions are provided for transforming scalar data appearing in complex-conjugate pairs into real-valued data, as explained in [4, Section 3.5].

For more details type `help rkfit`. Some of the capabilities of RKFUN are shown in the following section.

## 6 The RKFUN class

The `rkfun` class is the fundamental data type to represent and work with rational functions. It has already been described above how to evaluate an `rkfun` object for scalar and matrix arguments by calling `ratfun(z)` or `ratfun(A,v)`, respectively. There are more than 20 other methods implemented for `rkfun`, and a list of all these can be obtained by typing `methods rkfun`. Here we provide a complete list with brief descriptions.

<code>basis</code>	- Orthonormal rational basis functions of an <code>rkfun</code> .
<code>coeffs</code>	- Expansion coefficients of an <code>rkfun</code> .
<code>contfrac</code>	- Convert an <code>rkfun</code> into continued fraction form.
<code>diff</code>	- Differentiate an <code>rkfun</code> .
<code>disp</code>	- Display information about an <code>rkfun</code> .
<code>double</code>	- Convert an <code>rkfun</code> into double precision (undo <code>vpa</code> or <code>mp</code> ).
<code>ezplot</code>	- Easy-to-use function plotter.
<code>feval</code>	- Evaluate an <code>rkfun</code> at scalar or matrix arguments.
<code>hess</code>	- Convert an <code>rkfun</code> pencil to (strict) upper-Hessenberg form.
<code>inv</code>	- Invert an <code>rkfun</code> corresponding to a Moebius transform.
<code>isreal</code>	- Returns true if an <code>rkfun</code> is real-valued.
<code>minus</code>	- Scalar subtraction.
<code>mp</code>	- Convert an <code>rkfun</code> into Advanpix Multiple Precision format.
<code>mrdivide</code>	- Scalar division.
<code>mtimes</code>	- Scalar multiplication.
<code>plus</code>	- Scalar addition.
<code>poles</code>	- Return the poles of an <code>rkfun</code> .
<code>poly</code>	- Convert an <code>rkfun</code> into a quotient of two polynomials.
<code>power</code>	- Integer exponentiation of an <code>rkfun</code> .
<code>rdivide</code>	- Division of two <code>rkfuns</code> .
<code>residue</code>	- Convert an <code>rkfun</code> into partial fraction form.
<code>rkfun</code>	- The <code>rkfun</code> constructor.
<code>roots</code>	- Compute the roots of an <code>rkfun</code> .
<code>size</code>	- Returns the size of an <code>rkfun</code> .
<code>subsref</code>	- Evaluate an <code>rkfun</code> (calls <code>feval</code> ).
<code>times</code>	- Multiplication of two <code>rkfuns</code> .
<code>type</code>	- Return the type (m+k,m) of an <code>rkfun</code> .
<code>uminus</code>	- Unary minus.
<code>uplus</code>	- Unary plus.
<code>vpa</code>	- Convert <code>rkfun</code> into MATLAB's variable precision format.

The names of these methods should be self-explanatory. For example, `roots(ratfun)` will return the roots of a `ratfun`, and `residue` will compute the partial fraction form. Most methods support the use of MATLAB's Variable Precision Arithmetic (VPA) or the Advanpix Multiple Precision toolbox (MP). So, for example, `contfrac(mp(ratfun))` will compute a continued fraction expansion of `ratfun` using multiple precision arithmetic. For more details on each of the methods, type `help rkfun.<method name>`. The RKFUN gallery provides some predefined rational functions that may be useful. A list of the options can be accessed as follows:

```
help rkfun.gallery
```

**GALLERY**      Collection of rational functions.

`obj = rkfun.gallery(funname, param1, param2, ...)` takes `funname`, a case-insensitive string that is the name of a rational function family, and the family's input parameters.

See the listing below for available function families.

<code>constant</code>	Constant function of value <code>param1</code> .
<code>cheby</code>	Chebyshev polynomial (first kind) of degree <code>param1</code> .
<code>cayley</code>	Cayley transformation $(1-z)/(1+z)$ .
<code>moebius</code>	Moebius transformation $(az+b)/(cz+d)$ with <code>param1 = [a,b,c,d]</code> .
<code>sqrt</code>	Zolotarev sqrt approximation of degree <code>param1</code> on the positive interval <code>[1,param2]</code> .
<code>sign</code>	Zolotarev sign approximation of degree <code>2*param1</code> on the union of <code>[1,param2]</code> and <code>[-param2,-1]</code> .
<code>step</code>	Unit step function approximation for <code>[-1,1]</code> of degree <code>2*param1</code> with steepness <code>param2</code> .

Another way to create an RKFUN is to make use of MATLAB's symbolic engine. For example, `r = rkfun('(x+1)*(x-2)/(x-3)^2')` will create a rational function as expected. Alternatively, one can specify a rational function by its roots and poles (and an optional scaling factor) using the `rkfun.nodes2rkfun` function. For example, `r = rkfun.nodes2rkfun([-1,2],[3,3])` will create the same rational function as above.

## 7 References

- [1] Advanpix LLC., *Multiprecision Computing Toolbox for MATLAB*, version 3.9.4.10481, Tokyo, Japan, 2016. <http://www.advanpix.com/>.
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